MATH 1220 Review for FINAL EXAM

6.3 (6) $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \le t \le 3$

$(2,5)$ to $(e^3+e^{-3}, -1)$

$\frac{dx}{dt} = e^t - e^{-t}$

$\frac{dy}{dt} = -2$

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\int \sqrt{\left(e^{2t} - 2 + e^{-2t}\right) + 4}$

$\int \sqrt{\left(e^t + e^{-t}\right)^2}$

$\int_0^3 \left(e^t + e^{-t}\right) dt$

$e^t - e^{-t}\Big]_0^3 = \boxed{e^3 - e^{-3}}$

(8) $y = \frac{x^2}{2} - \frac{1}{4}\ln x \quad 2 \le x \le 4$

$\frac{dy}{dx} = x - \frac{1}{4x}$

$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$\int_2^4 \sqrt{1 + \left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)}\, dx$

$\int_2^4 \left(x + \frac{1}{4x}\right) dx = \frac{1}{2}x^2 + \frac{1}{4}\ln x\Big]_2^4 = \left(8 + \frac{\ln 4}{4}\right) - \left(2 + \frac{\ln 2}{4}\right)$

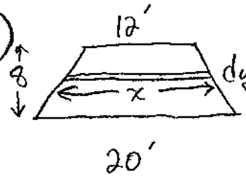
$= \boxed{6 + \frac{\ln 2}{4}}$

6.4 (4) $h(r) = \frac{3}{(1+r)^2}$ $[1,6]$

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx$

$h_{ave} = \frac{1}{6-1}\int_1^6 \frac{3}{(1+r)^2}\, dr = \frac{1}{5}\left[-\frac{3}{1+r}\right]_1^6 = \frac{1}{5}\left[-\frac{3}{7} + \frac{3}{2}\right] = \boxed{\frac{3}{14}}$

6.5 (25)



Depth

x	y
12	0
20	8

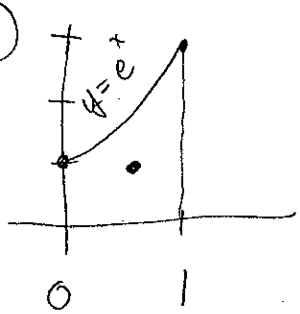
$x = y + 12$

$F = PA = \delta d A$

$F = \int_0^8 62.5\, y \left[\overset{x}{(y+12)}\, dy\right] = \frac{104{,}000}{3}$

$\approx \boxed{34{,}667 \text{ lbs}}$

(37)



$m = \rho\int_0^1 e^x\, dx = (e-1)\rho$

$M_x = \rho\int_0^1 \frac{1}{2}\left(e^x\right)^2 dx = \left(\frac{e^2-1}{4}\right)\rho$

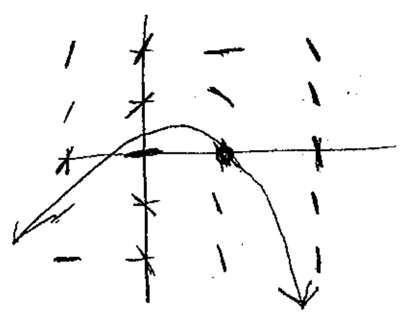
$M_y = \rho\int_0^1 x e^x\, dx = \rho$

$\bar{x} = \frac{M_y}{m} = \frac{1}{e-1} \approx .58$

$\bar{y} = \frac{M_x}{m} = \frac{e+1}{4} \approx .930$

7.2 (11) $y' = y - 2x$ $(1,0)$

x	-1	-1	-1	-1	-1	0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
y	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2	-2	-1	0	1	2
y′	0	1	2	3	4	-2	-1	0	1	2	-4	-3	-2	-1	0	-6	-5	-4	-3	-2



7.3 10 $\boxed{\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2} \qquad y(0)=1}$

$$\int \frac{1+y^2}{y}\,dy = \int \cos x\,dx$$

$$\ln|y| + \tfrac{1}{2}y^2 = \sin x + C$$

$$\boxed{\ln|y| + \tfrac{1}{2}y^2 = \sin x + \tfrac{1}{2}}$$

$$\ln(1) + \tfrac{1}{2}(1)^2 = \sin(0) + C$$

$$\tfrac{1}{2} = C$$

7.4 3 $P(t) = P_0 e^{rt}$

$420 = 100e^{r(1)}$

$4.2 = e^r$

$\ln 4.2 = r \approx 1.435$

a) $P = 100e^{1.435t}$

b) $P(3) = 100e^{1.435(3)} \approx \boxed{7409}$

c) $\left.\dfrac{dP}{dt} = 143.5e^{1.435t}\right|_{t=3}$ $\quad \dfrac{dP}{dt} \approx 10{,}632 \dfrac{\text{Bacteria}}{\text{Hour}}$

d) $10{,}000 = 100e^{1.435t}$

$\dfrac{\ln 100}{1.435} = t \approx \boxed{3.209 \text{ Hours}}$

8.1 10 $\boxed{a_n = \dfrac{n+1}{3n-1}} = \dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \ldots$

Converges $\displaystyle\lim_{n\to\infty} \frac{n+1}{3n-1} = \boxed{\frac{1}{3}}$

12 $\boxed{a_n = \dfrac{n^{1/2}}{1+n^{1/2}}} = \dfrac{\sqrt{1}}{1+\sqrt{1}}, \dfrac{\sqrt{2}}{1+\sqrt{2}}, \dfrac{\sqrt{3}}{1+\sqrt{3}}, \dfrac{\sqrt{4}}{1+\sqrt{4}}, \ldots$

Converges $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \boxed{1}$

8.2 13 $\boxed{\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}} = 5 + \dfrac{10}{3} + \dfrac{20}{9} + \dfrac{40}{27} + \cdots \quad \left(r = \dfrac{2}{3}\right)$

$$S_\infty = \frac{a_1}{1-r} = \frac{5}{1-\frac{2}{3}} = \boxed{15}$$

8.2 (20) $\boxed{\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}} = \frac{1(3)}{4^2} + \frac{2(4)}{5^2} + \frac{3(5)}{6^2} + \frac{4(6)}{7^2} + \cdots$

$\boxed{\text{Divergent}}$ (Test for Divergence)

$$\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = 1$$

8.3 (7) $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4}}$ $\int_1^{\infty} \frac{1}{x^4}\,dx = -\frac{1}{3}x^{-3}\Big]_1^{\infty} = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$

CONVERGENT

(26) $\boxed{\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}}$ $\quad b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{1}{n^{4/3}}$ [Convergent p-series]

$\underbrace{}_{a_n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1+\frac{5}{n}}{\sqrt[3]{1+\frac{1}{n^5}}} = 1 \Rightarrow$$

$\boxed{a_n \text{ Converges (Limit Comparison Test)}}$

8.4 (13) $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad |R_n| < .00005}$

$$\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \cdots$$

$\boxed{n=5 \text{ Terms will yield desired accuracy}}$

(23) $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}} = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \cdots$ Alt Series is Convergent (Alt Series Test)

$b_n = |a_n| = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ (Divergent p-series)

a_n is NOT absolutely convergent

8.5 (4) $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}}$

Ratio Test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right| = \frac{n+1}{n+2}|x| \Rightarrow \lim_{n\to\infty} = |x|$$

Want $|x| < 1$

$\boxed{(-1, 1]}$

$x = -1$

$a_n = \frac{1}{n+1}$ Compare to $\frac{1}{n}$ DIV (Limit Comparison)

$x = 1$

$a_n = \frac{(-1)^n}{n+1}$

CONV (Alt Series)

8.7 (9) $f(x) = 1 + x + x^2 \quad a = 2$

$f(2) = 7$

$f'(x) = 1 + 2x \qquad f'(2) = 5$

$f''(x) = 2 \qquad f''(2) = 2$

$f'''(x) = 0 \qquad f'''(2) = 0$

Taylors Series

$$\frac{7}{0!}(x-2)^0 + \frac{5}{1!}(x-2)^1 + \frac{2}{2!}(x-2)^2 + 0 + \dots$$

$$\boxed{7 + 5(x-2) + 1(x-2)^2}$$

8.8 (3) $$\frac{1}{(2+x)^3} = \frac{1}{2^3\left(1+\frac{x}{2}\right)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n$$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

$$= \frac{1}{8}\left[1 + \frac{-3}{1!}\left(\frac{x}{2}\right)^1 + \frac{-3(-4)}{2!}\left(\frac{x}{2}\right)^2 + \frac{-3(-4)(-5)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right]$$

$$= \frac{1}{8} + \frac{1}{2^3}\sum_{n=1}^{\infty}(-1)^n\frac{(n+2)!\ x^n}{n!\,2\cdot 2^n}$$

$$= \frac{1}{8} + \sum_{n=1}^{\infty}(-1)^n\frac{(n+2)!\ x^n}{n!\ 2^{n+4}} = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{(n+1)(n+2)}{2^{n+4}}x^n}$$

9.1 (16) Endpoints of Diameter $(2,1,4)$ & $(4,3,10)$ — A, B

Midpoint Formula give center of $(3,2,7)$ — C

$r = AC = \sqrt{1^2+1^2+3^2} = \sqrt{11}$

$$\boxed{(x-3)^2 + (y-2)^2 + (z-7)^2 = 11}$$

9.2 (19) $\vec{v} = \langle 8,-1,4\rangle \qquad |\vec{v}| = \sqrt{64+1+16} = 9 \qquad \vec{u} = \frac{1}{9}\vec{v} = \boxed{\left\langle \frac{8}{9}, -\frac{1}{9}, \frac{4}{9}\right\rangle}$

9.3 (13) $\vec{a} = \langle -8,6\rangle \quad \vec{b} = \langle\sqrt{7},3\rangle$

$\vec{a}\cdot\vec{b} = |a|\,|b|\cos\theta$

$-8\sqrt{7} + 18 = (10)(4)\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{18-8\sqrt{7}}{40}\right) \approx \boxed{94.54^\circ}$

9.4 (14) $\left.\begin{array}{l}\vec{v} = \langle 1,1,1\rangle \\ \vec{w} = \langle 2,0,1\rangle\end{array}\right\}$ $\vec{v}\times\vec{w} = \langle 1-0, 2-1, 0-2\rangle \xrightarrow{\text{Normalize}} \frac{1}{\sqrt{6}}\langle 1,1,-2\rangle \Rightarrow \begin{cases}\vec{u}_1 = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right\rangle \\ \vec{u}_2 = \left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle\end{cases}$

9.5 (6) Line through $(6,1,-3)$ & $(2,4,5)$ — P, Q

$\vec{v} = \vec{PQ} = \langle -4,3,8\rangle$

$x = 2-4t$
$y = 4+3t$
$z = 5+8t$

OR

$$\frac{x-2}{-4} = \frac{y-4}{3} = \frac{z-5}{8}$$

9.6 (21) $4x^2 + y^2 - 4y + 4z^2 - 24z = -36$

$4x^2 + y^2 - 4y + 4 + 4(z^2 - 6z + 9) = -36 + 4 + 36$

$4x^2 + (y-2)^2 + 4(z-3)^2 = 4$

$$\boxed{\frac{x^2}{1} + \frac{(y-2)^2}{4} + \frac{(z-3)^2}{1} = 1}$$

Ellipsoid centered at $(0,2,3) \quad r_x = 1 \quad r_y = 2 \quad r_z = 1$

Vertices at $(-1,2,3)$ & $(1,2,3)$; $(0,0,3)$ & $(0,4,3)$; $(0,2,2)$ & $(0,2,4)$

9.6 (23) $x^2+y^2=1$ a) in $\mathbb{R}^2$ represents the UNIT CIRCLE

b) in $\mathbb{R}^3$ it represents a CYLINDER (radius of 1, z-axis through center)

c) $x^2+z^2=1$ is a CYLINDER (radius of 1, y-axis through center)

AH-1 (18) $\boxed{x+y=9} \Longrightarrow r\cos\theta + r\sin\theta = 9 \Rightarrow r(\sin\theta+\cos\theta)=9 \Rightarrow \boxed{r=\frac{9}{\sin\theta+\cos\theta}}$

AH-2 (21) $\boxed{r_1=3\cos\theta \quad r_2=1+\cos\theta}$ Find area inside the first, outside the second.

$r_1=r_2$

$3\cos\theta=1+\cos\theta$

$2\cos\theta=1$

$\cos\theta=1/2$

$\theta=-\pi/3$ OR $\theta=\pi/3$

$$A=\int_a^b \frac{1}{2}r^2\,d\theta$$

$$A=\int_{-\pi/3}^{\pi/3}\frac{1}{2}\left[(3\cos\theta)^2-(1+\cos\theta)^2\right]d\theta = \int_0^{\pi/3}(9\cos^2\theta-2\cos\theta-1)\,d\theta = \boxed{\pi}$$

9.7 (76) $\boxed{(2, \pi/3, \pi/4)}$
$\rho \quad \theta \quad \phi$

$x=2\sin(\pi/4)\cos(\pi/3) = 2(\sqrt{2}/2)1/2 = \sqrt{2}/2$

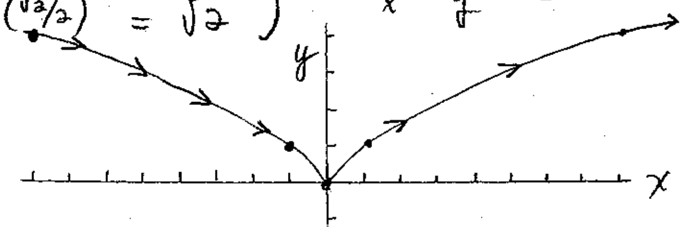
$y=2\sin(\pi/4)\sin(\pi/3) = 2(\sqrt{2}/2)\sqrt{3}/2 = \sqrt{6}/2$

$z=2\cos(\pi/4) = 2(\sqrt{2}/2) = \sqrt{2}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$
x y z

10.1 (6) $\boxed{r(t)=\langle t^3, t^2\rangle}$

t	-2	-1	0	1	2
x	-8	-1	0	1	8
y	4	1	0	1	4



10.2 (35) $\boxed{r'=\langle 2t, 3t^2, \sqrt{t}\rangle \quad r(1)=\langle 1,1,0\rangle}$ $r=\int r'\,dt = \langle t^2+c_1, t^3+c_2, \frac{2}{3}t^{3/2}+c_3\rangle$

$\boxed{r(t)=\langle t^2, t^3, \frac{2}{3}t^{3/2}-\frac{2}{3}\rangle}$ From intial conditions: $1^2+c_1=1$; $1^3+c_2=1$; $\frac{2}{3}(1)^{3/2}+c_3=0$

$c_1=0$ $c_2=0$ $c_3=-2/3$

10.3 (1) $\boxed{r(t)=\langle 2\sin t, 5t, 2\cos t\rangle \quad -10\le t\le 10}$

$r'=\langle 2\cos t, 5, -2\sin t\rangle$

$|r'|=\sqrt{4\cos^2 t+25+4\sin^2 t}=\sqrt{29}$

$$L=\int_{-10}^{10}\sqrt{29}\,dt = \boxed{20\sqrt{29}}$$

10.3 (19) $\boxed{r(t)=\langle t, t^2, t^3\rangle \text{ Find } K \text{ at } (1,1,1),\ t=1}$

$r'=\langle 1, 2t, 3t^2\rangle$

$r''=\langle 0, 2, 6t\rangle$

$r'\times r''=\langle 6t^2, -6t, 2\rangle$

at $t=1$

$r'\times r''=\langle 6, -6, 2\rangle$

$r'=\langle 1,2,3\rangle$

$$K(t)=\frac{|r'\times r''|}{|r'|^3}; \quad K(1)=\frac{\sqrt{36+36+4}}{(\sqrt{1+4+9})^3}=\frac{2\sqrt{19}}{14\sqrt{14}}=\frac{1}{7}\sqrt{\frac{19}{14}} \text{ or } \frac{\sqrt{266}}{98}$$

10.4 (14) $\boxed{\vec{a}=\langle 2, 6t, 12t^2\rangle \quad \vec{v}(0)=\langle 1,0,0\rangle \quad \vec{r}(0)=\langle 0,1,-1\rangle}$

$\vec{v}(t)=\int\vec{a}(t)\,dt = \langle 2t+c_1, 3t^2+c_2, 4t^3+c_3\rangle$

$c_1=1$, $c_2=0$, $c_3=0$ (see given initial conditions)

$\vec{v}(t)=\langle 2t+1, 3t^2, 4t^3\rangle$

$\vec{r}(t)=\int\vec{v}(t)\,dt = \langle t^2+t+c_4, t^3+c_5, t^4+c_6\rangle$

$c_4=0$, $c_5=1$, $c_6=-1$

$\vec{r}(t)=\langle t^2+t, t^3+1, t^4-1\rangle$

SPEED $= v(t)=|\vec{v}(t)| = \sqrt{(2t+1)^2+(3t^2)^2+(4t^3)^2}$

$v(t)=\sqrt{16t^6+9t^4+4t^2+4t+1}$